Spring 2008 Lecture 3

LINEAR ELASTICITY

Introduction:

In this lecture, we will learn about elastic deformations. In an elastic deformation, the body returns to its original shape when the load is removed.

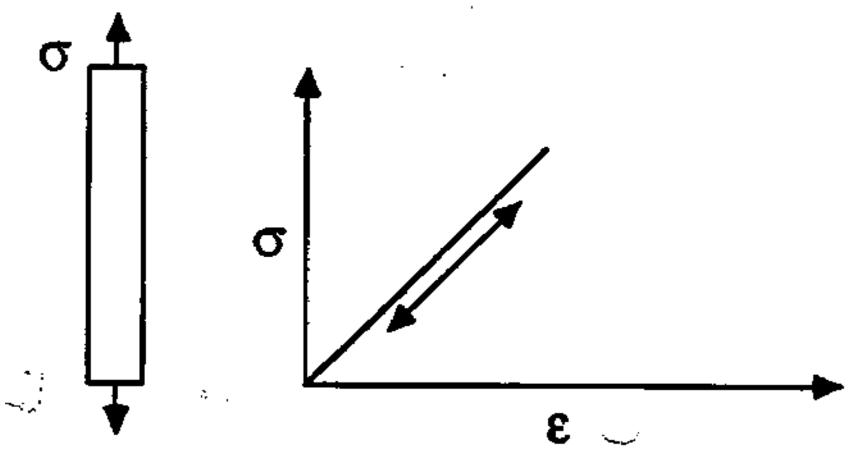

Figure 1: A linear elastic deformation. The material returns to the original shape upon removal of the load. The stress/strain relation is linear for both loading and unloading.

Figure 1 shows the simplest case of a linear elastic deformation where in addition to returning to the original shape upon removal of the load, the stress/strain relation is linear. Not all elastic deformations are linear. Figure 2shows the typical non-linear large elastic deformation of rubber. However, note that at small strains the stress/strain relation is linear.

We will now review the elastic material properties that define the linear elastic stress/strain relation of materials.
The linear elastic stress/strain relation is only valid in the regime of small strains. To simplify the notation, we will not distinguish the difference between engineering e and true strain e and the strain components will be denoted with e.

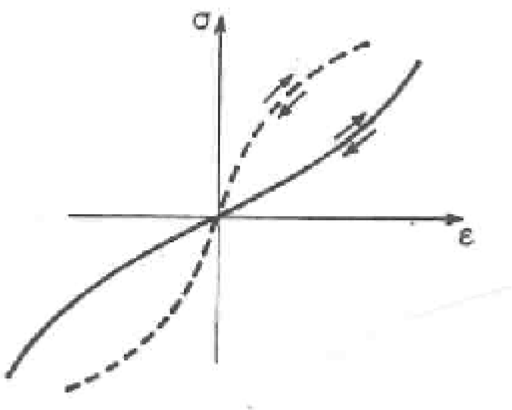

Figure 2: A non-linear elastic stress/strain curve for rubber. Note that at small strains the relation can be approximated as linear.

Young's Modulus E:

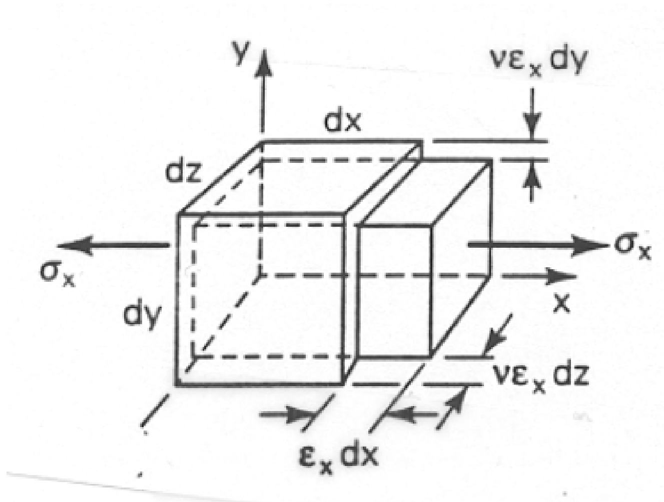

Figure 3: A uniaxial tensile experiment in the direction x.

Consider a uniaxial loading in the direction x. Figure 3 shows the deformation of the specimen. We define the Young's Modulus E as the ratio of the imposed normal stress σ to the induced normal strain e in the direction of the stress, i.e.

$$E = \frac{\sigma}{\epsilon} \tag{1}$$

Note that in this uniaxial test, $\sigma = \sigma_{xx}$ and all other stress components are equal to zero.
Also, we here denote e = e_{xx} but note that the strains e_{yy} and e_{zz} are not zero!

Equation (1) is known as the one-dimensional Hooke's law.

Poisson's Ratio ν :

Let us again consider the uniaxial test shown in Fig. 3. We define the Poisson's ratio ν as follows:

$$\nu = -\frac{\epsilon_{yy}}{\epsilon_{xx}} = (\text{for an isotropic solid}^1) \ -\frac{\epsilon_{zz}}{\epsilon_{xx}} \tag{2}$$

where e_{xx} is the strain in the direction of the uniaxial stress and e_{yy} and e_{zz} are the (negative) strains in the transverse directions.

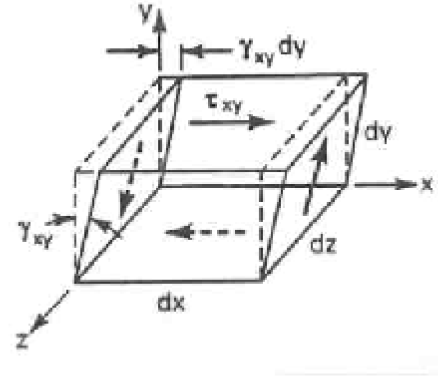

Figure 4: A shear test.

Shear Modulus G:

The shear modulus G is defined for the state of shear shown in Figure 4 as follows:

$$\gamma = \frac{1}{G}\tau \tag{3}$$

Note that for an isotropic material, G is not a new independent property. We will later be

$$G = \frac{E}{2(1+\nu)} \tag{4}$$

Here, we concentrate on isotropic solids, i.e. solids that have the same properties in all directions.

Generalized Hooke's Law:

Let us assume that all nine stress components are acting at the same time in the body. Using superposition of the induced strains when each stress component is acting alone, we can write the linear elastic equations as follows:

$$\epsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right], \quad \gamma_{xy} = \frac{\tau_{xy}}{G} \tag{5}$$

$$\epsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\right], \quad \gamma_{yz} = \frac{\tau_{yz}}{G} \tag{6}$$

$$\epsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right], \quad \gamma_{xz} = \frac{\tau_{xz}}{G} \tag{7}$$

Bulk Modulus, B:

In lecture 2, we expressed the relative change in volume $\Delta = \Delta_V/V$ in terms of the normal strain components. Using the definition of the hydrostatic stress $\sigma_m = (\sigma_{xx} + \sigma_{yy} + \sigma_{zz})/3$ introduced in lecture 2, $\Delta = e_{xx} + e_{yy} + e_{zz}$ and expressing the strain components in terms of the stress components via Hooke's law, you should be able to easily show that:

$$\Delta = \frac{\Delta V}{V} = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} == \frac{1-2\nu}{E}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right) = \left[\frac{1}{B}\right]\sigma_m \tag{8}$$

where the bulk modulus B was defined as:

$$B = \frac{E}{3(1-2\nu)} \tag{9}$$

Note that for the case of Fig. 5 ($\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p$), equ. (9) leads to the following:

$$\Delta = \frac{\Delta V}{V} = -\left[\frac{1}{B}\right]p \tag{10}$$

The bulk modulus B thus defines the linear elastic relation between the relative change in volume, *($\Delta V/V$)*, of the material and the applied hydrostatic pressure p.

Based on the above result we conclude that elastic deformations are incompressible when $\nu = \frac{1}{2}$ (which for most materials is not the case!).
<u>Exercise 1:</u> Using the generalized Hooke's law, show that $G = E/2(1+\nu)$

(Hint: Consider the plane stress state corresponding to $\sigma_{xx} = -\sigma_{yy}$ with all other stress components zero. This stress state was examined in lecture 1where it was shown to be equivalent to pure shear!)

Figure 5: Hydrostatic pressure $\sigma_{11} = \sigma_{22} = \sigma_{33} = -p$ results in change of volume and not in distortion of the cube.

Effective elastic modulus in plane strain problems:

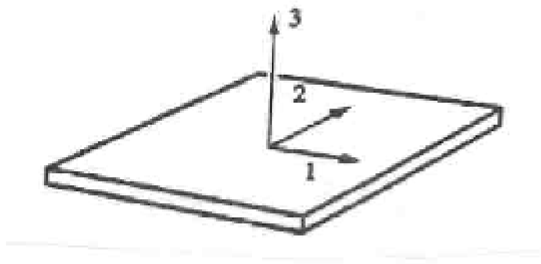

Figure 6: Plane strain ($e_{22} = 0$) in a thin plane ($\sigma_{33} = 0$).

The plane shown in Figure 6 is loaded in tension along direction 1, but is prevented from contracting along direction 2(we consider the axes 1, 2and 3 to be principal axes). Using $e_2 = 0$ (as specified), $\sigma_3 = 0$ (free surface) and Hooke's generalized law, we can show that:

$$\epsilon_2 = \frac{\sigma_2}{E} - \nu\frac{\sigma_1}{E} = 0, \text{ so } \sigma_2 = \nu\sigma_1 \tag{11}$$

$$\epsilon_1 = \frac{\sigma_1}{E} - \nu\frac{\sigma_2}{E} = \frac{\sigma_1}{E}\left(1 - \nu^2\right) \tag{12}$$

The effective elastic modulus E' in the direction 1 can now be defined as follows:

$$E' = \frac{\sigma_1}{\epsilon_1} = \frac{E}{1 - \nu^2} \tag{13}$$

Using the concept of effective elastic modulus allows us to now treat the problem as one-dimensional (in direction 1). However, one needs to be careful as it is not always possible to simplify this way most deformation problems.

Elastic work (strain energy density):

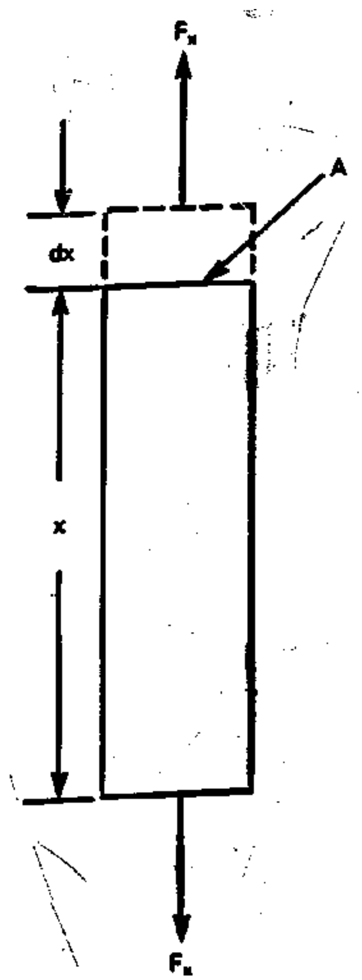

Figure 7: The tensile test in the direction *x*. The current length is denoted with *x* and the increment in length as *dx*.

The incremental work *dW* for a uniaxial tension test (see Figure 7) is given as:

$$dW = F_x dx \tag{14}$$

and the work per unit volume (strain energy density) is:

$$dw = \frac{F_x dx}{Ax} = \sigma_{xx} d\epsilon_{xx} \tag{15}$$

and with integration (recall from Hooke's 1D law, $\sigma_{xx} = Ee_{xx}$)

$$w = \frac{1}{2}\sigma_{xx}\epsilon_{xx} = \frac{1}{2}E\epsilon_{xx}^2 \tag{16}$$

In the general three dimensional stress state, the elastic work per unit volume can be calculated as follows (use superposition!):

$$w = \frac{1}{2}\left(\sigma_{xx}\epsilon_{xx} + \sigma_{yy}\epsilon_{yy} + \sigma_{zz}\epsilon_{zz} + \sigma_{xy}\gamma_{xy} + \sigma_{yz}\gamma_{yz} + \sigma_{zx}\gamma_{zx}\right) \tag{17}$$

The elastic work can also be written in terms of principal stresses and strains as follows:

$$w = \frac{1}{2}\left(\sigma_1\epsilon_1 + \sigma_2\epsilon_2 + \sigma_3\epsilon_3\right) \tag{18}$$

Components of strain energy:

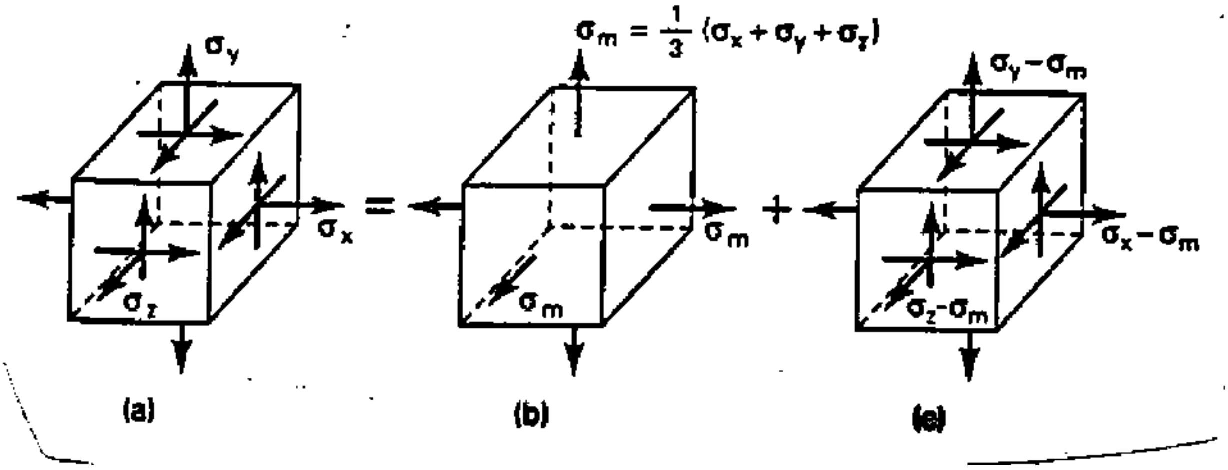

Figure 8: Decomposition of (a) state of stress into (b) dilatational stresses and (c) distortional or deviatoric stresses.

A new perspective on strain energy may be gained by viewing the general state of stress as the superposition of the hydrostatic (dilatational) stresses and the deviatoric (distortional) stresses (see Fig. 8 and lecture 2).
The hydrostatic stress state (Part (b) of Fig. 8) results in volume change without distortion. Associated with σ_m is the dilatational (mean) strain

$$\epsilon_m = \frac{1}{E}\left(\sigma_m - \nu\left(\sigma_m + \sigma_m\right)\right) = \frac{1-2\nu}{E}\sigma_m = \frac{\sigma_m}{3B} \tag{19}$$

The dilatational strain energy absorbed per unit volume is given as

$$W_v = 3\frac{1}{2}\sigma_m\epsilon_m = \frac{\sigma_m^2}{2B} = \frac{\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)^2}{18B} = \frac{\left(\sigma_1 + \sigma_2 + \sigma_3\right)^2}{18B} \tag{20}$$

The deviatoric stress state (Part (c) of Fig. 8) produces distortion without change in volume. The distortional energy per unit volume, W_d, is attributable to the change of shape of the unit volume while the volume remains constant. To calculate Wd, use the strain energy equation (17) with all strain components expressed in terms of stresses (via Hooke's law) and subtract the dilatational strain energy given in eq. (20). We finally arrive at the following expression:

$$W_d = \frac{1}{12G}\left[\left(\sigma_{xx}-\sigma_{yy}\right)^2+\left(\sigma_{yy}-\sigma_{zz}\right)^2+\left(\sigma_{zz}-\sigma_{xx}\right)^2+6\left(\tau_{xy}^2+\tau_{xz}^2+\tau_{yz}^2\right)\right] \qquad (21)$$

W_d can also be expressed in terms of principal stresses as follows:

$$W_d = \frac{1}{12G}\left[\left(\sigma_1-\sigma_2\right)^2+\left(\sigma_2-\sigma_3\right)^2+\left(\sigma_3-\sigma_1\right)^2\right] \qquad (22)$$

If you are wondering what is the big deal with the above decomposition, you will find later in this course that the most popular 'yield criterion' (for transition from elastic toplastic (permanent) deformations) is based on *Wd* taking a critical value (which is a materialproperty).