Spring 2008
Lecture 1

THE CONCEPT OF STRESS

Uniaxial stress definition:

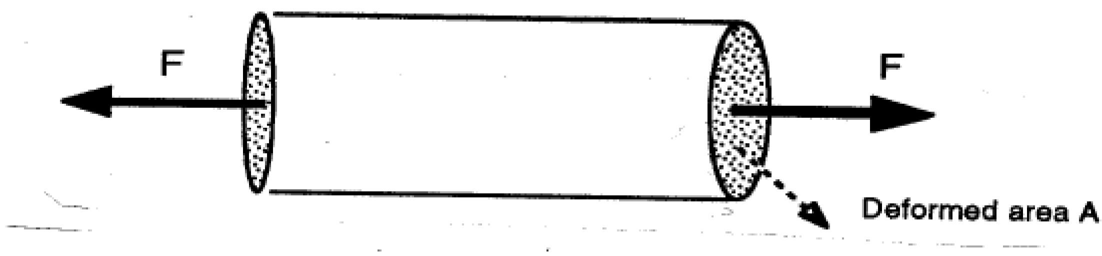

Figure 1: A uniaxial tensile test. A force F is applied perpendicular to the area A. Before the application of the force, the cross-section area was A_o.

The uniaxial true stress σ is defined as follows:

$$\sigma = \frac{F}{A} \tag{1}$$

The engineering or nominal stress is also defined as force/unit area, where the original area (before the application of the force) is taken:

$$\sigma_0 = \frac{F}{A_0} \tag{2}$$

The relation between the two definitions can be easily derived as follows:

$$\sigma_o = \frac{F}{A_o} = \frac{F}{A}\frac{A}{A_o} = \sigma\frac{A}{A_o} \tag{3}$$

Stress definition in three dimensions:

We want to define the stress at a point O in a continuous body loaded by external forces (see Fig. 2). The first step is to "conceptually" cut the body into two pieces across a plane that passes from the point O. Let **n** be the **unit vector** normal to the surface generated by the cut as shown in Fig. 3. Here we show only one of the pieces of the body that results from the above cut. The forces acting on the cut surface are the internal forces transmitted from the other piece and are necessary to maintain the two pieces in static equilibrium

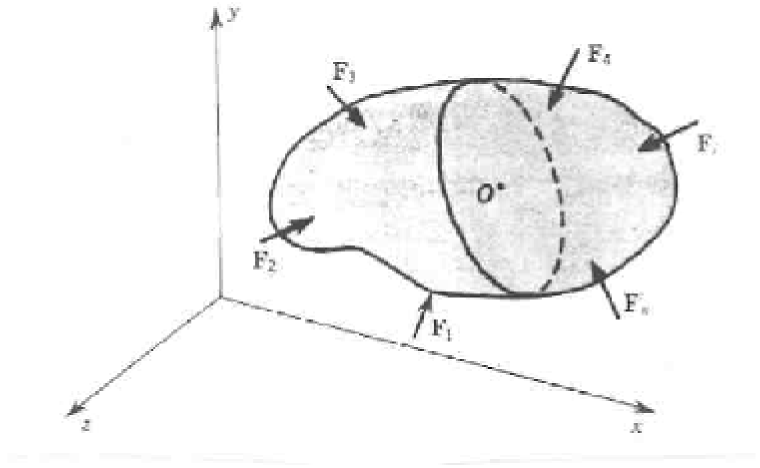

Figure 2: Continuous body acted on by external forces.

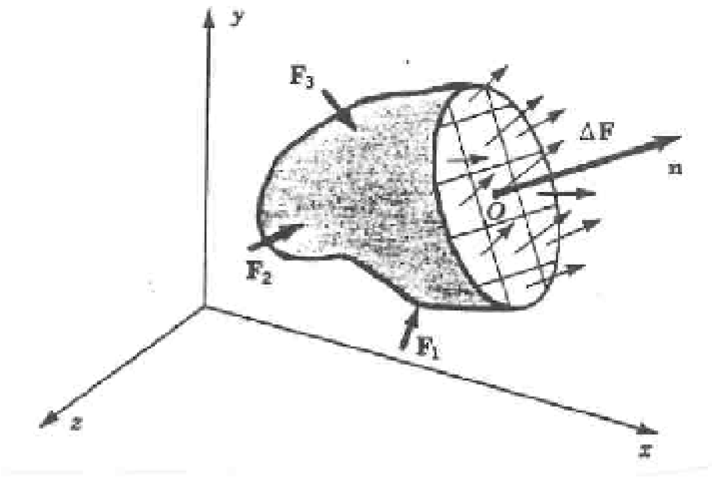

Figure 3: Internal forces acting on a plane whose normal is n.

For the plane cut shown, let us define the traction vector t_n as:

$$\mathbf{t_n} = \lim_{\Delta A \to 0} \frac{\Delta \mathbf{F}}{\Delta A} \quad (4)$$

where ΔF is the internal force acting in a small area ΔA around the point *O* (see Fig. 3). We see that t_n is 'force intensity' or 'stress' acting on a plane whose normal is **n** at the point *O*. If we consider a fixed Cartesian coordinate system x, y, z with unit vectors e_x, e_y, e_z, then, we can write the components of the traction vector t_n as follows:

$$\mathbf{t_n} = t_{nx}\mathbf{e}_x + t_{ny}\mathbf{e}_y + t_{nz}\mathbf{e}_z \quad (5)$$

We say that we know the state of stress at a point if for any plane passing through that point we can calculate the traction vector. Above we calculated the traction t_n at the point *O* through the plane with normal **n. It turns out that if we know the traction vector (force per unit area) in three mutually perpendicular planes through point *O*, then we can always calculate the traction vector at any other plane through** *O*.

Select **n** = e_x, e_y and e_z (unit vectors in the *x*, *y* and *z* axes, respectively). This defines three traction forces (t_{ex} , t_{ey} , t_{ez}) acting on the *yz*, *xz* and *xy* intersections per unit areas in the corresponding planes. Each of these traction forces has three components. In particular, we can write the following:

$$\mathbf{t}_{\mathbf{e}_x} = \sigma_{xx}\mathbf{e}_x + \sigma_{xy}\mathbf{e}_y + \sigma_{xz}\mathbf{e}_z \tag{6}$$

$$\mathbf{t}_{\mathbf{e}_y} = \sigma_{yx}\mathbf{e}_x + \sigma_{yy}\mathbf{e}_y + \sigma_{yz}\mathbf{e}_z \tag{7}$$

$$\mathbf{t}_{\mathbf{e}_z} = \sigma_{zx}\mathbf{e}_x + \sigma_{zy}\mathbf{e}_y + \sigma_{zz}\mathbf{e}_z \tag{8}$$

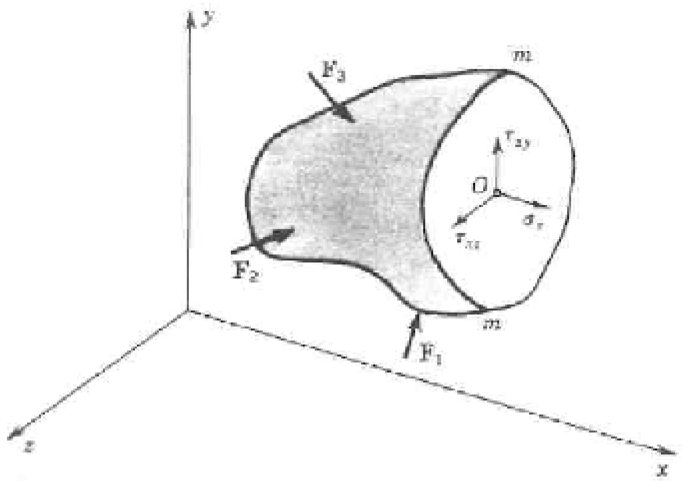

Figure 4: Stress components on positive x face at point *O*.

Equation 6 is graphically shown in Fig. 4 and similar representations are applied for equations 7 and 8. We define the stress components to be nothing else but the *x*, *y*, and *z* components of the traction vectors t_{ex} , t_{ey} , t_{ez} as given in equations (6), (7) and (8).

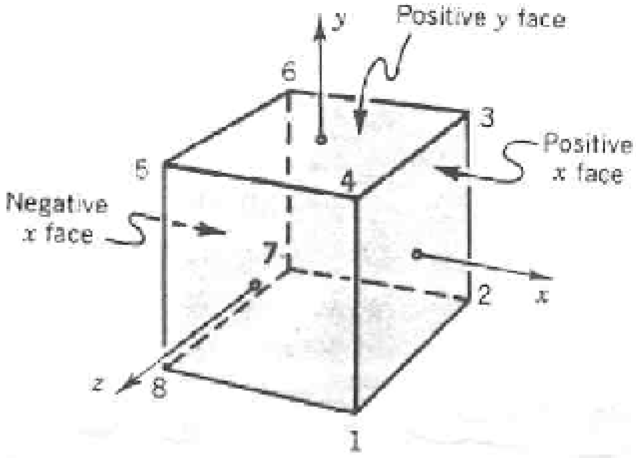

Figure 5: Definition of positive and negative cube faces.

In a matrix form, the "stress at the point *O*" is denoted as follows:

$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \tag{9}$$

We depict the three planes discussed above as the faces of an infinitesimal cube enclosing the point *O* with the axes *x, y* and *z* normal to the three faces of the cube. The face of the cube that faces in the +*x* direction is defined as a positive face of the cube. Similarly, we define the positive cube faces corresponding to the +*y* and +*z* directions (see Fig. 5). Let us return to the decomposition of equ. (6). The traction t_e is applied on the *yz* plane. Its component σ_{xx} is in the *x* direction (i.e. normal to the *yz* plane), while the components σ_{xy} and σ_{xz} are in the *y* and *z* directions, respectively (i.e. they lie in the plane *yz)*. We call the stress component σ_{xx} the <u>normal stress</u> component, while σ_{xy} and σ_{xz} are known as <u>shear stresses</u>. Similar terminology is applied to the other stress components. In general, a stress component σ_{ij} denotes the *j* component of the traction (force/unit area) that is applied in the area with outward positive normal the *i* axis. The definition of positive stress components is summarized in Fig. 6

To complete this section, we will show how you can calculate the traction t_n acting in any plane with normal n if the stress components in any coordinate system (here *x, y, z)* are known. The components t_{nx} t_{ny} and t_{nz} of the traction force t_n (see eq. 5) in any plane of unit normal n = $n_x e_x$ + $n_y e_y$ + $n_z e_z$ (see Figure 3) are given as:

$$\begin{Bmatrix} t_{nx} \\ t_{ny} \\ t_{nz} \end{Bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} \tag{10}$$

Note that *n1, n2* and *n3* are nothing else but the 'directional cosines' of the unit vector n.

Also using moment balance concepts, one can show that $\sigma_{xy} = \sigma_{yx}$, $\sigma_{yz} = \sigma_{zy}$ and $\sigma_{zx} = \sigma_{xz}$.

Finally, it is customary to denote the shear stresses using τ instead of σ. So keep in mind that where you see τ_{xy}, we mean σ_{xy}, etc. Also, for normal stresses it is quite usual not to use repeated indices. For example, σ_x is often used to denote σ_{xx}.

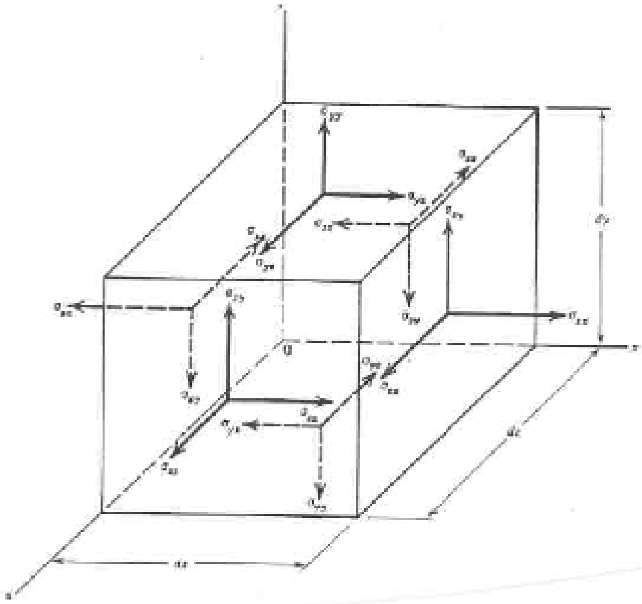

Figure 6: All stress components are considered positive as shown. In the positive cube planes, the stress components are positive if they point in the positive directions. In the negative cube faces, the stress components are positive if they point in the negative directions!

Principal stress components:

If you examine all possible sets of coordinate systems (in practice this means rotating the cube around the point *O*), then one can show that there is a set of 3 mutually perpendicular planes through the point *O* where all the shear stresses acting on the surfaces of the small cube are zero. Let us denote these special axes as 1, 2 and 3. We call them the principal stress directions and the normal stress components acting in the planes 23, 13 and 12 the principal σ_1, σ_2 and σ_3 stress components, respectively.
Using the principal stress directions, the stress at the point *O* can be written in a matrix form as follows:

$$\sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \tag{11}$$

It is important to understand that the x, y, z stress components (eq. 10) or the principal stress components (eq. 11) contain the same information about the stress at the point *O* (i.e. knowing the stress components σ_{xx}, σ_{yy}, σ_{zz}, σ_{xy}, σ_{yz} and σ_{zx}, you can always find σ_1, σ_2 and σ_3 and the directions of the principal axes - also knowing the directions of the principal stress axes and the values of the principal stresses one can recover all stress components in any x, y, z coordinate system).

However, it is always easier to work in a coordinate system where there are no shear components (the algebra will be much simplified!). Figure 7 presents some examples of stress states in terms of principal stress components.
How can we find the principal stress directions and the principal stresses? We will present the

answer to this question shortly.

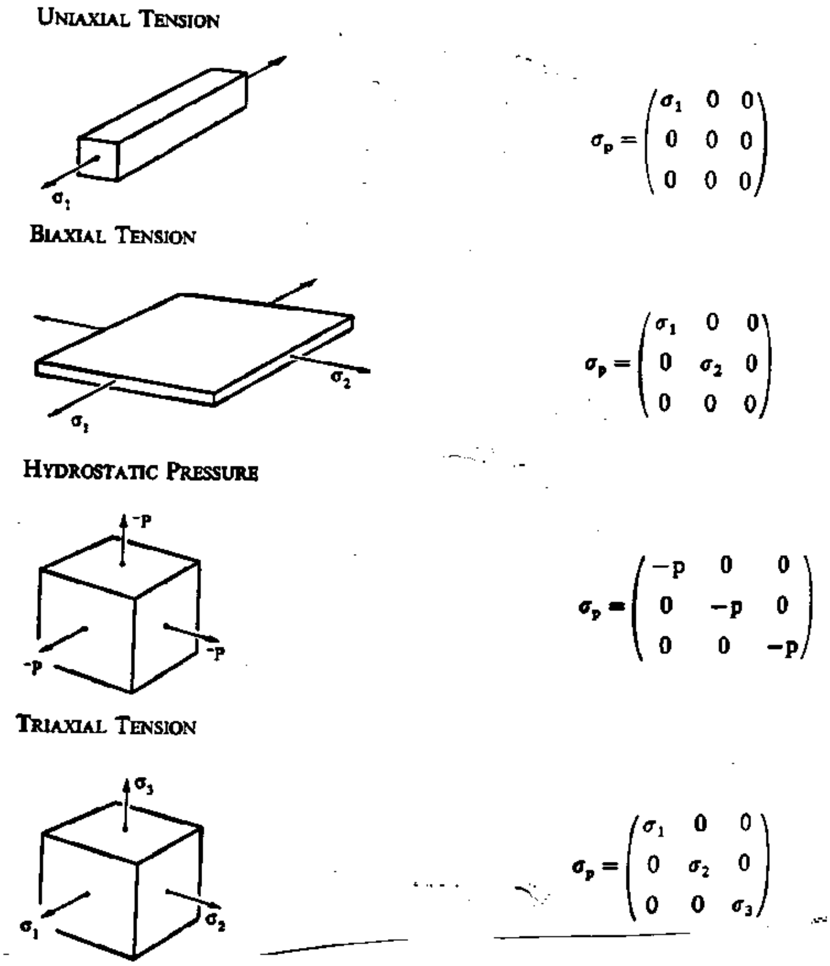

Figure 7: Examples of stress states in terms of principal stress components.

Calculation of stress components from one coordinate system to another for plane stress problems:

Consider the *xy* plane to be the plane of a thin sheet. The state of stress at a given point can be approximated to depend only upon the four stress components σ_{xx}, σ_{xy}, σ_{yx}, and σ_{yy} that are considered to be functions of only the *x* and *y*. The stress components $\sigma_{zz} = \sigma_{zx} = \sigma_{zy} = 0$, i.e. all the out of the *xy* plane stress components are zero. Since the only non zero stress components are on the *xy* plane, we call this state plane stress in the *xy* plane. (Note that in terms of principal stresses, plane stress means that one of the principal stresses is zero).

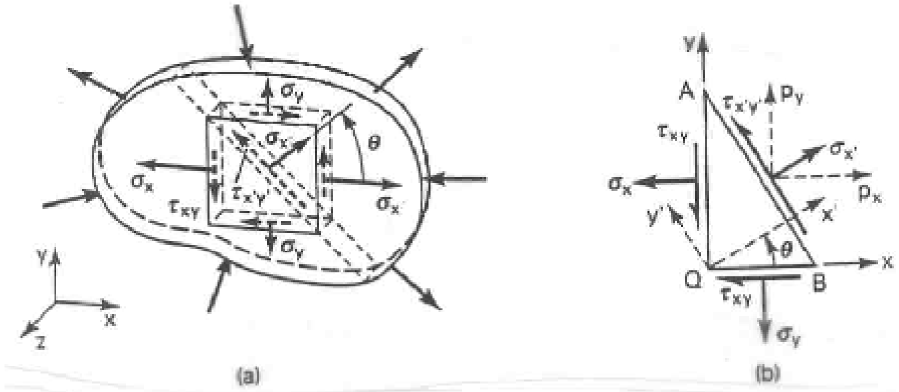

Figure 8: Elements in plane stress. p_x and p_y are here the x and y components, respectively, of the traction vector in the plane AB.

Consider a square element (the plane version of the cube) in plane stress (see Fig. 8). Let us consider a coordinate system x' and y' (on the plane xy) such that θ is the angle between the x' axis and the x axis. From equilibrium of forces in the triangular QAB (Fig. 8), it can be shown that:

$$\sigma_{x'x'} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \tag{12}$$

$$\tau_{x'y'} = -\frac{\sigma_{xx} - \sigma_{yy}}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \tag{13}$$

To calculate σ_{yy}, you can use equ. 12 with $\theta + \pi/2$ instead of θ. Finally:

$$\sigma_{y'y'} = \frac{\sigma_{xx} + \sigma_{yy}}{2} - \frac{\sigma_{xx} - \sigma_{yy}}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \tag{14}$$

Recall that in the above plane stress state the z axis is a principal stress axis. To define the other two principal directions in the xy plane, we can use eq. (13) with τ_x>$_y$> = 0 to calculate the principal angle θ.
From $T_x i_y i$ =0, one can show that the direction of principal axes is given from:

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}} \tag{15}$$

while the principal in plane stresses are given as follows:

$$\sigma_{1,2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} \tag{16}$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} \tag{17}$$

The calculation of principal stresses in three dimensional stress states is slightly more complicated and it will not be needed in this course.
To find the maximum in plane shear stress (i.e. the maximum shear stress in the plane *xy)* take $^{d\tau}$3- =0 (use equ. (13)), find the angle *9* and then substitute back into equ. (13). We finally can show the following:

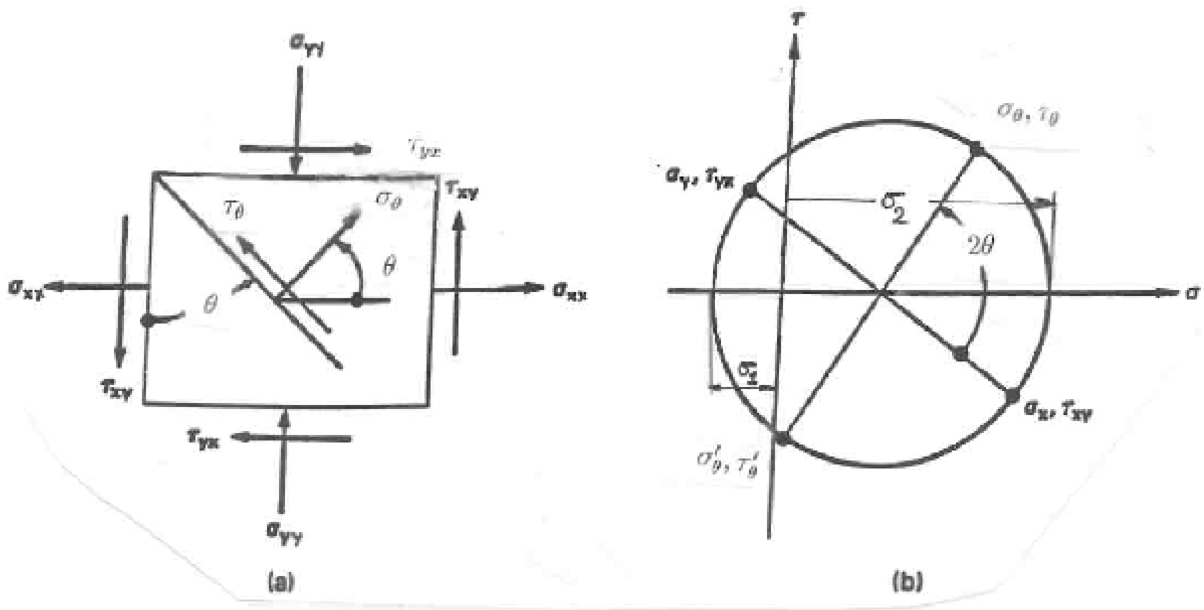

Figure 9: (a) Stress element and (b) Mohr's circle for a plane stress state.

For the above plane stress state, it is simple to use the so called Mohr Circle (Fig. 9) to perform stress transformations. The basic rules for using the Mohr circle are as follows:

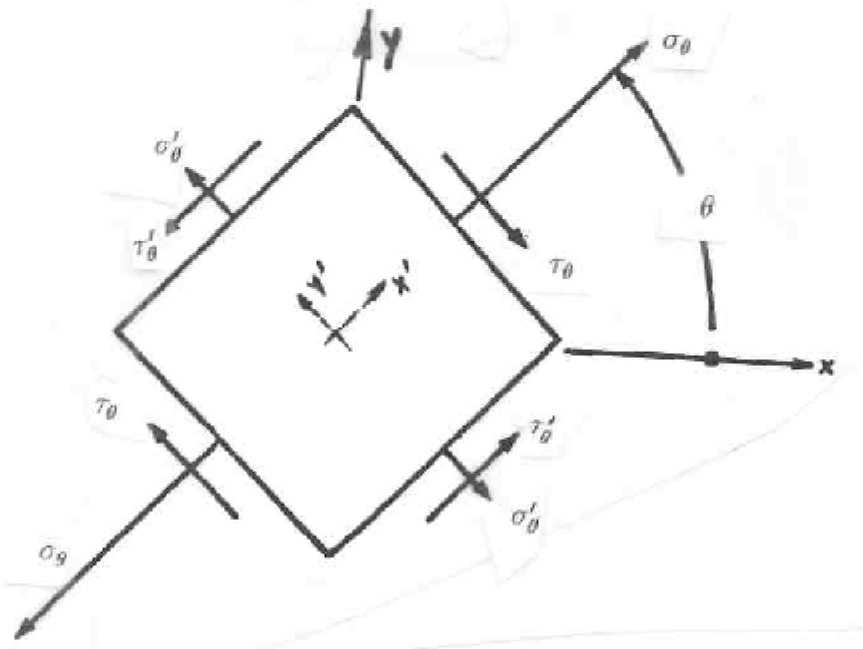

Figure 10: The Mohr circle provides both normal stresses$a_{x}/_{x}/ = o_\theta$ and $a_y>y> = o_\theta$ as well as the shear stress $r_x>_y> = T_\theta$.

Normal stresses are plotted to scale along the abscissa, where tensile is positive and compressive is negative (Figures 9 and 10).

1. Shear stresses are plotted along the ordinate. A shear stress that induces clockwise rotation of the stress element is plotted as if it were positive, while one causing counterclockwise rotation as if it were negative.

2. Angles between planes or directions represented on the circle plot are double the corresponding angles on the physical plane.

Using the design of Fig. 9, try to validate equations (13), (15), (16) and (17).

Calculation of the maximum shear stress at a point using the principal stress values:

Above we calculated the maximum in plane shear stress but not necessarily the maximum shear stress in all planes! Let the principal stresses be ordered as follows:$\sigma_1 \geq \sigma_2 \geq \sigma_3$. Then the maximum shear stress τ_{max} in any plane through the point is given as:

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} \tag{18}$$

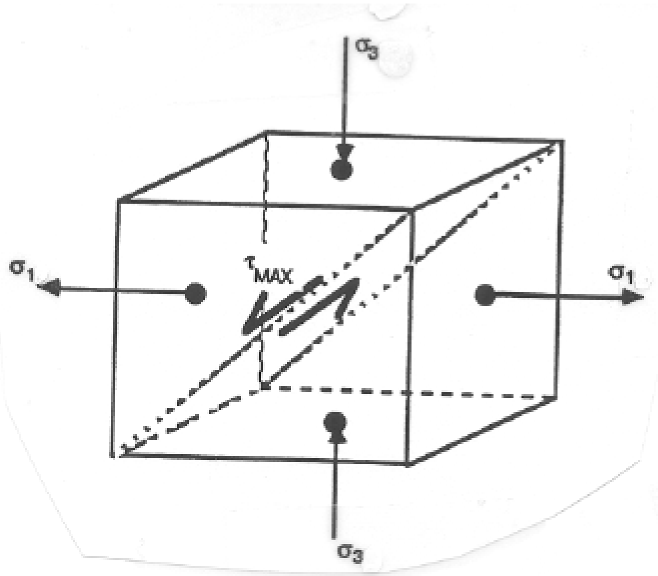

Figure 11: The maximum shear stress at a point in terms of the principal stresses.

The maximum shear stress acts on a plane that makes an angle of 45^o degrees with the planes in which the principal stresses σ_1 and σ_3 act (see Fig. 11).
Note that in eq. (18) the algebraic sign for shear stresses is maintained, i.e. if a principal stress is compressive, then a negative value will be entered in its value in eq. (18).

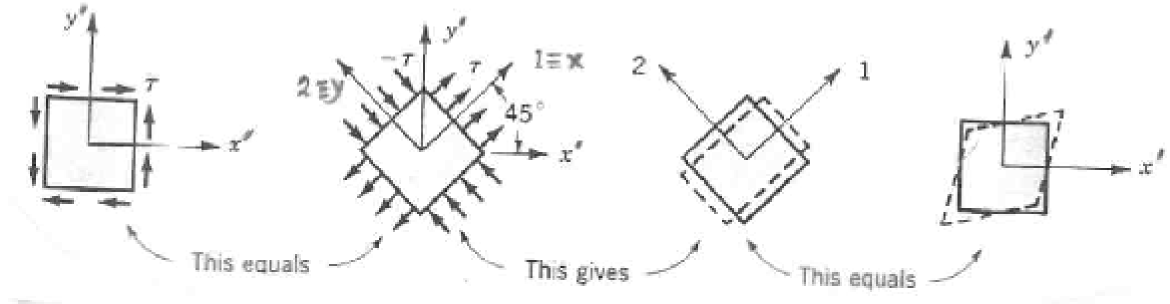

Figure 12: The plane stress state σ_{xx} = —σ_{yy} the 45*o* planes with respect to the *x* and *y* axes Let us consider a simple example of plane stress where $\sigma_{xx} = -\sigma_{yy} = \sigma > 0$ and $\sigma_{xy} = 0 = \sigma$ and $\sigma_{zz} = 0$ that leads to pure shear in (see Fig. 12). Using (18), we can write:

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma - (-\sigma)}{2} = \sigma \tag{19}$$

i.e. the maximum shear stress is equal to σ and is applied in the planes 45^o from the *x* and *y* axes. The above biaxial stress state is often used to simulate a shear state and as such it has both analytical and experimental use.

Stress equilibrium equations in two-dimensional problems:

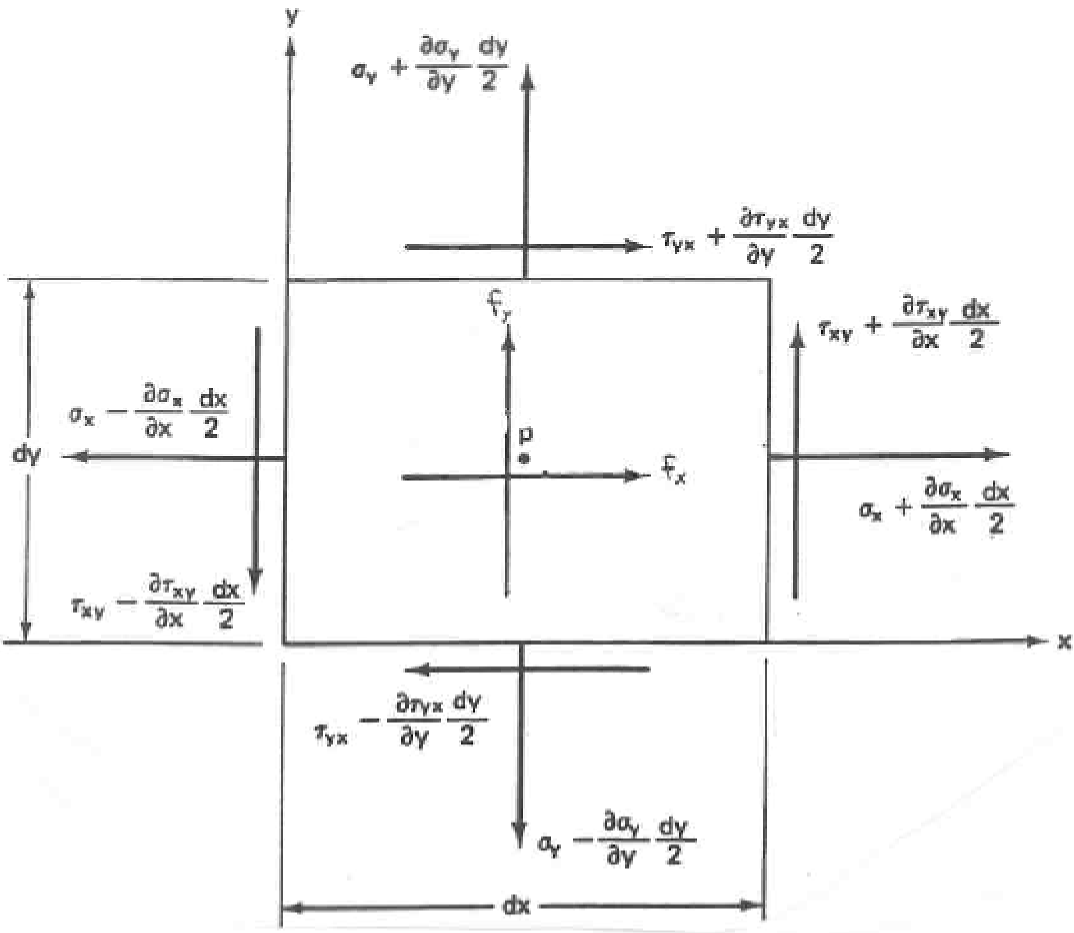

Figure 13: Stress equilibrium in an infinitesimal element under plane stress.

Let us consider an infinitesimal square element of sizes *dx* x *dy* (the two-dimensional version of the cube) under plane stress conditions. Let σ_{xx}, τ_{xy}, and σ_{yy} be the stress components at the center of the square. Assuming a continuous variation of the stresses (i.e. that the stress components vary from point to point in a continuous way), we can approximate the values of the stress components at the four edges of the square element using a Taylor's series expansion (see Fig. 13). Also, let f_x and f_y be the body forces per unit area (or per unit volume if you think of the thickness being equal to 1) applied in the square section (e.g., gravity forces).

Equilibrium of forces in the *x* direction results in the following equation:

$$\sum F_x = 0 \text{ or } \sum \ (\text{stress}) \times \ (\text{area}) \ = 0 \tag{20}$$

or

$$\left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\frac{dx}{2}\right) dy + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\frac{dy}{2}\right) dx - \left(\sigma_{xx} - \frac{\partial \sigma_{xx}}{\partial x}\frac{dx}{2}\right) dy - \left(\tau_{yx} - \frac{\partial \tau_{yx}}{\partial y}\frac{dy}{2}\right) dx + f_x dx dy = 0$$

Note that we have here assumed that the stress components are functions of *x* and *y* and that σ_{xx}, σ_{yy} and τ_{xy} are the values of these functions at the center of the square. Similarly the derivatives - f^{3}^, etc. are computed at the center of the square. In this sense it will

have been more accurate in the equation above to write $\sigma_{xx}(0,0)$ instead of σ_{xx}, etc. and $\frac{\partial}{\partial}{}^{\sigma}{}_{x}$^(0,0) instead of ^g^{3}^, etc. However, we suppress all these details and we only highlight

the process of derivation as well as the final results.

Expanding and rearranging the equation above gives the following

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + f_x = 0 \tag{21}$$

Following the same approach for $\sum F_y = 0$ gives

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + f_y = 0 \tag{22}$$

The above two equations were derived for the center of the particular cube but it should be obvious by now that these equations should be valid at any point of the body.

The above derivations are very essential for analyzing deformation problems. In the remaining of this course we will not use the equilibrium equations in their present form. However, when we discuss forming processes, we will consider similar calculations to the above and derive stress equilibrium equations appropriate to the conditions of each of the forming process examined.